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HYPOELLIPTICITY OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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Abstract

In this paper, we study the hypoellipticity problems for fully nonlinear partial differential equations of order m. For a solution $u\in C^{\rho}_{loc}(\Omega)$, if the linearized operator for the nonlinear equation on u satisfies some subelliptic conditions, we can deduce $u\in C^{\infty}(\Omega)$ by using the paradifferential operator theory of J.-M. Bony.

§0 Introduction

Let us consider the following equation:

$$F[u]=F(x,u(x),\cdots,\partial^{\beta}u(x),\cdots)_{|\beta|\leq m}=0 \tag{0.1}$$

where $x\in\Omega$, $\Omega\subset\mathbb{R}^n$ open, F is a real C^{∞} function.

If $u\in C^{\rho}_{loc}(\Omega)$ $(\rho>m)$ is a real solution for the equation (0.1), we define an associate linearized operator:

$$P(x,D)=\sum_{|\alpha|>2m-\rho} a_\alpha(x)\,\partial_x^\alpha \tag{0.2}$$

where $a_\alpha(x)=\frac{\partial F}{\partial u_\alpha}(x,u(x),\cdots,\partial^\beta u(x),\cdots)\in C_{loc}^{\rho-m}(\Omega)$. Its symbol

$$p(x,\xi)=\sum_{|\alpha|>2m-\rho} a_\alpha(x)(i\xi)^\alpha. \tag{0.3}$$

Then we obtain the following main theorem.

<u>Theorem 0.1.</u> Suppose that $u\in C_{loc}^{\rho}(\Omega)$ is a real solution of equation (0.1), $0\leqslant m'\leqslant m$, $0\leqslant\delta<\frac{1}{2}$ and $\rho>m+1+\frac{1}{1-2\delta}(m-m')$, and the symbol defined by (0.3) satisfies:

H$_1$) $\forall K\subset\subset\Omega$, $\exists R>0$, $C_1>0$, $C_2>0$, s.t.

$$C_1|\xi|^{m'}\leqslant|p(x,\xi)|\leqslant C_2|\xi|^m,$$

$$\forall x\in K,\ \xi\in\mathbb{R}^n,\ |\xi|\geqslant R.$$

H$_2$) $\forall K\subset\subset\Omega$, $\forall\alpha,\beta\in\mathbb{N}^n$, $|\beta|<\rho-m$, $\exists R>0$, $C_{\alpha,\beta,K}>0$ s.t.

$$|\partial_x^\beta\partial_\xi^\alpha p(x,\xi)|\leqslant C_{\alpha,\beta,K}|p(x,\xi)||\xi|^{-|\alpha|+\delta|\beta|},$$

$$\forall x\in K,\ \xi\in\mathbb{R}^n,\ |\xi|\geqslant R.$$

Then $u\in C^\infty(\Omega)$.

§1 Nonhomogeneous symbolic calculus

First we recall that for any constants $0<\varepsilon_1<\varepsilon_2<1$, $R>0$, there exists a function $\psi(\eta,\xi)\in C^\infty(\mathbb{R}^n\times\mathbb{R}^n)$, such that $\psi=0$ if $|\eta|\geqslant\varepsilon_2|\xi|$, $\psi=1$ if $|\eta|\leqslant\varepsilon_1|\xi|$ and $|\xi|\geqslant R$, and for any $\alpha,\beta\in\mathbb{N}^n$, there exists a constant $C_{\alpha,\beta}>0$, such that

$$|\partial_\eta^\alpha \partial_\xi^\beta \psi(\eta,\xi)| \leq C_{\alpha,\beta}(1+|\xi|)^{-|\alpha|-|\beta|}. \tag{1.1}$$

Definition 1.1 For constants $0 \leq \delta < 1$, $m \in \mathbb{R}$, $r > 0$ $(r \notin \mathbb{Z})$, we define the symbol space

$$\Sigma_{r,\delta}^m = \left\{ p(x,\xi) \,\middle|\, \begin{array}{l} \text{defined on } \mathbb{R}^n \times \mathbb{R}^n,\ C^\infty \text{ in } \xi,\ C^r \text{ in } x; \\ \forall \alpha, \beta \in \mathbb{N}^n,\ |\alpha| < r,\ \exists C_{\alpha,\beta} > 0,\ \text{s.t.} \\ |\partial_x^\alpha \partial_\xi^\beta p(x,\xi)| \leq C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|+\delta|\alpha|},\ \forall \xi \in \mathbb{R}^n. \end{array} \right\}.$$

$$\sigma_p(x,\xi) = (2\pi)^{-n} \int e^{i\eta x} \psi(\eta,\xi) \hat{p}(\eta,\xi)\, d\eta.$$

where $\psi(\eta,\xi)$ is a cut-off function in (1.1).

We have the following properties for the symbol class defined above.

Proposition 1.2 If $p(x,\xi) \in \Sigma_{r,\delta}^m$, then $\sigma_p(x,\xi) \in S_{1,1}^m$.

Proposition 1.3 (Composition of symbols)

Let $p \in \Sigma_{r,\delta}^m$, $q \in \Sigma_{r,\delta}^{m'}$ $(r > 1)$, then

(1) $$p \# q = \sum_{|\alpha| < r-1} \frac{1}{\alpha!} \partial_\xi^\alpha p(x,\xi) D_x^\alpha q(x,\xi) \in \Sigma_{r-[r]+1,\delta}^{m+m'}.$$

(2) $$\sigma_{p\#q} - \sigma_p \# \sigma_q \in S_{1,1}^{m+m'-(1-2\delta)[r]},$$ where

$$\sigma_p \# \sigma_q = \sum_{|\alpha| < r-1} \frac{1}{\alpha!} \partial_\xi^\alpha \sigma_p(x,\xi) D_x^\alpha \sigma_q(x,\xi).$$

§2 The proof of Theorem 0.1

By the para-linearization process, one can find the following para-linearization theorem in [6, 12].

Theorem 2.1 Let $u \in C_{loc}^\rho(\Omega) \cap H_{loc}^s(\Omega)$, $\rho > m$, $s > 0$, be a real solution for the equation (0.1), $P(x,D)\in Op(\Sigma^m_{\rho-m}(\Omega))$ is the paradifferential operator whose symbol $\sigma(P)=p(x,\xi)$ is defined by (0.3). Then there exists a function $f\in C^{2\rho-2m}_{loc}(\Omega)\cap H^{s+\rho-2m}_{loc}(\Omega)$ such that $Pu=f$.

Theorem 2.2 Let $u\in C^\rho_{loc}(\Omega)\cap H^s_{loc}(\Omega)$ $(s>0)$ satisfy the assumptions of Theorem 0.1, then there exists a constant $\varepsilon>0$ (independent of s), such that $u\in C^\rho_{loc}(\Omega)\cap H^{s+\varepsilon}_{loc}(\Omega)$.

The proof of Theorem 0.1:

From the fact that $u(x)\in C^\rho_{loc}(\Omega)$ and the assumptions of Theorem 0.1, we can deduce $u(x)\in H^{m+1}_{loc}(\Omega)$. From Theorem 2.2, we know $u(x)\in C^\rho_{loc}(\Omega)\cap H^{m+1+\varepsilon}_{loc}(\Omega)$. By induction, repeating the process k times, we can obtain $u(x)\in C^\rho_{loc}(\Omega)\cap H^{m+1+k\varepsilon}_{loc}(\Omega)$. This implies $u(x)\in C^\rho_{loc}(\Omega)\cap\bigcap_{k=1}^{+\infty}H^{m+1+k\varepsilon}_{loc}(\Omega)$. Finally by the Sobolev embedding theorem, we know $\bigcap_{k=1}^{+\infty}H^{m+1+k\varepsilon}_{loc}(\Omega)=C^\infty(\Omega)$. Thus we have proved $u(x)\in C^\infty(\Omega)$.

§3 An example

Let us consider

$$F[u]=(\log u)^{16}\left(\frac{\partial^4 u}{\partial x^4}+\frac{\partial^4 u}{\partial y^4}\right)-\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)-2u(\log u)^{16}+2u=0,$$

where $(x,y)\in\Omega\subset\mathbb{R}^2$, $(0,0)\in\Omega$.

Conclusion: Suppose $u(x,y)\in C^\rho_{loc}(\Omega)$, $u(x,y)>0$, $\rho>9$ is a real solution for the equation above, then $u(x,y) \in C^{\infty}(\Omega)$.

In fact, $u(x,y)=e^{x+y}$ is a solution for the equation, but at $(0,0)$, the linearized operator is degenerately elliptic.

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